Wednesday, April 1, 2026 1:24 PM

Q1 Finding Minors and Cofactors

Let $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$

Sol

$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 40 - 24 = 16$

minor, R, C

$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 \cdot 16 = 16$

$M_{12} = 10$ $\quad M_{13} = 3$ $\quad M_{21} = +24$

$M_{22} = 28$ $\quad M_{23} = 11$ $\quad M_{31} = 26$

$M_{32} = 26$ $\quad M_{33} = 13$

$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 \, 10 = -10$

$C_{13} = (-1)^{1+3} M_{13} = (-1)^4 \, 3 = 3$

$C_{21} = (-1)^{2+1} \, 24 = -24$

$C_{22} = (-1)^4 \, 28 = 28$

$C_{23} = (-1)^5 \, 11 = -11$

$C_{31} = (-1)^4 \, 26 = 26$

$C_{32} = (-1)^5 \, 26 = -26$

$C_{33} = (-1)^6 \, 13 = 13$

Cofactor Expansion along the first Row

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix} \quad \text{find } \det(A) = ?$$

Sol//

$$\Rightarrow +(+3)\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (+1)\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + (+0)\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$\Rightarrow 3(8-12) - 1(4-15) + 0(-8+20)$$

$$\Rightarrow 3(-4) - 1(-11) + 0$$

$$\Rightarrow -12 + 11 = -1$$

2nd Row

$$\Rightarrow -(-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 0 \\ 5 & -2 \end{vmatrix} - (+3)\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix}$$

$$\Rightarrow 2(-2-0) - 4(-6-0) - 3(12-5)$$

$$-4 + 24 - 21$$

$$-25 + 24 = -1$$

3rd Row

$$\Rightarrow 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix} - 4\begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix} \cancel{+} - 2\begin{vmatrix} 3 & 1 \\ -2 & -4 \end{vmatrix}$$

$$\Rightarrow 5(3-0) - 4(9-0) - 2(-12+2)$$

$$15 - 36 + 20 = 35 - 36 = -1$$

$15 \quad -36 \quad +20 \quad = \quad 35-36 = -1$

Cofactor Expansion along the First Column

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix} \qquad \det(A) = ?$$

$$\Rightarrow 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + (5)\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$\Rightarrow 3(8-12) \quad +2(-2-0) \quad +5(3-0)$$

$$-12 \quad -4 \quad +15 \quad = \quad -16+15 = -1$$

Smart Choice of Row or Column

$$\text{let } A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix} \qquad \det(A) = ??$$

Sol we use Cofactor expansion along 2nd Column

$$\det(A) = 1 \begin{vmatrix} 1 & 0 & -1 \end{vmatrix}$$

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

We use Cofactor expansion along 2nd Column

$$= 1 \cdot (-2) \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \Rightarrow -2(1-(-2)) \Rightarrow -2(3) = -6$$

Using Row Reduction to Evaluate a determinant

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

R_{12}

$$\det(A) = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} = -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$R_3 - 2R_1 = -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \Rightarrow R_3 - 10R_2 \; (-3) = \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 10 & -5 \end{vmatrix} \qquad \begin{vmatrix} 0 & 0 & -55 \end{vmatrix}$$

$$\Rightarrow (-3)(-55)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow \text{upper triangular matrix}$$

$$\Rightarrow (-3)(-55)(1) = 165 \quad \text{Ans}$$

Evaluate determinant using Column Operations

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix} \qquad \det(A) = ?$$

$\neq 0$!?

Sol: using elementary Row operation we Reduce A into Row Echelon form (lower triangular matrix)

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{vmatrix}$$

multiply $-3C_1 + C_4$

$\det(A) = (1)(7)(3)(-26) = -546$

Row Operations and Cofactor Expansion

Evaluate $\det(A) = ?$

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

Sol

$$\det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} \quad \begin{matrix} R_1 - R_4 \\ 2R_2 - R_3 \\ \vdots \end{matrix}$$

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$ Cofactor Expansion along first column

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix}$$ use Cofactor expansion along first column

$= (-)(-1) \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix}$

$= (1)(9 - 27) = -18$ Az

find the Cofactors

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

$C_{11} = 12$ $\quad C_{12} = 6$ $\quad C_{13} = -16$

$C_{21} = 4$ $\quad C_{22} = 2$ $\quad C_{23} = 16$

$C_{31} = 12$ $\quad C_{32} = -10$ $\quad C_{33} = 16$

$$\text{Cofactors of } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$\text{Adj } A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$ find $Adj(A)$

Sol

$$Adj(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$ Ans

find the inverse of matrix A.

$$A^{-1} = \frac{1}{\det(A)} * Adj(A)$$

$$= \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 12/64 & 4/64 & 12/64 \\ 6/64 & 2/64 & -10/64 \\ -16/64 & 16/64 & 16/64 \end{bmatrix}$$ Ans

Cramer's Rule

Using Cramer's Rule solve the linear system

$$x_1 + 2x_3 = 6$$
$$-3x_1 + 4x_2 + 6x_3 = 30$$
$$-x_1 - 2x_2 + 3x_3 = 8$$

for x_1 $A_1 \to$ replace B, for x_2 $A_2 \to$ replace B, for x_3 A_3 replace B

Sol

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix}$$

Coefficient matrix A

$$\det(A) = \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix}$$

$$= 1(12+12) - 0(-9+6) + 2(6+4)$$
$$= 24 + 20 = 44$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11} = x_1$$

replace with B — remains same

$$\det A_1 = \begin{vmatrix} 6 & 0 & 2 \\ \ldots \end{vmatrix} = 6(12+12) - 0(90-48) +$$

$$\det A_1 = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} = 6(12+12) - 0(90-48) + 2(-60-32)$$

$$= 6(24) - 0 + 2(-92)$$

$$= 144 - 184$$

$$= -40$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11} = x_2$$

$$\begin{array}{ccc} \text{same} & \text{replace with } B & \text{same} \end{array}$$

$$\det A_2 = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix} = 1(90-48) - 6(-9+6) + 2(-24+30)$$

$$= 42 + 18 + 12$$

$$= 72$$

$$x_3 = \frac{\det(A_3)}{\det A} = \frac{152}{44} = \frac{38}{11} = x_3$$

$$\begin{array}{ccc} \text{same} & & \text{replace with } B \end{array}$$

$$\det(A_3) = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix}$$

$$= 1(32+60) - 0(\ ) + 6(6+4)$$

$$92 + 60 = 152$$