CSC2206 : Linear Algebra

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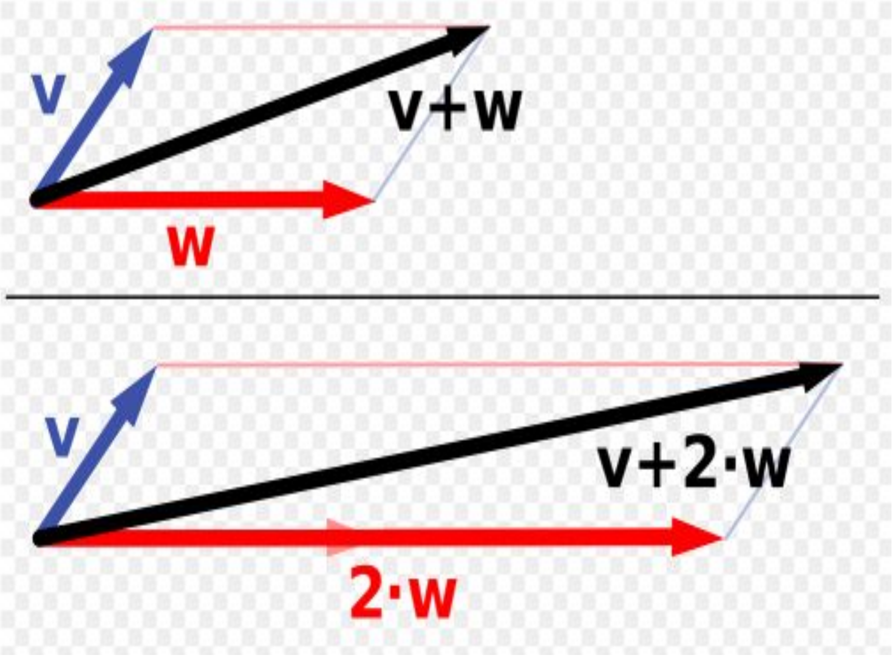
Overview

- Vector Spaces
- Subspaces of Vector Spaces
- Spanning Sets
- Linear Dependence and Independence
- Basis and Dimension
- Null space, Column space, Row space of a Matrix
- Rank of a Matrix and Systems of Linear Equations

Vector Space

- A **vector space** is a mathematical structure formed by a collection of vectors: objects that may be added together and multiplied ("scaled") by numbers, called scalars in this context.



Vector addition : a vector **v** (blue) is added to another vector **w** (red).

scalar multiplication: **w** is stretched by a factor of 2, yielding the sum **v** + 2·**w**.

Vector Space (Contd.)

- A ***vector space*** is defined as a *nonempty set V* of entities called ***vectors*** and associated scalars that satisfy the conditions outlined in Condition A and B in following slides.

- For example; with any two vectors **a** and **b** in ***V*** all their linear combinations $\boxed{\alpha_1 a + \alpha_2 b \ (\alpha_1, \alpha_2 \ \textit{any real numbers})}$ are elements of V

- A vector space is *real* if the scalars are real numbers; it is *complex* if the scalars are complex numbers.

Vector Space (Contd.)

- Condition A: There is in V i.e. vector space an operation called ***vector addition***, denoted $\boldsymbol{x} + \boldsymbol{y}$, that satisfies following rules:

1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for all vectors $\mathbf{x}$ and $\mathbf{y}$ in the space.
2. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ for all $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.
3. There exists in V a unique vector, called the *zero vector*, denoted by $\mathbf{0}$, such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ and $\mathbf{0} + \mathbf{x} = \mathbf{x}$ for all vectors $\mathbf{x}$.
4. For each vector $\mathbf{x}$ in V, there is a unique vector in V, called the *negation* of $\mathbf{x}$, and by denoted $-\mathbf{x}$, such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$ and $(-\mathbf{x}) + \mathbf{x} = \mathbf{0}$.

Vector Space (Contd.)

- Condition B: There is in V an operation called *multiplication by a scalar* that associates with each scalar c and each vector $\mathbf{x}$ in V a unique vector called the *product* of c and $\mathbf{x}$, denoted by $c\mathbf{x}$ and $\mathbf{x}c$, and which satisfies:

1. $c(d\mathbf{x}) = (cd)\mathbf{x}$ for all scalars c and d, and all vectors $\mathbf{x}$.
2. $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$ for all scalars c and d, and all vectors $\mathbf{x}$.
3. $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$ for all scalars c and all vectors $\mathbf{x}$ and $\mathbf{y}$.
4. For every x in V

$$1\mathbf{x} = \mathbf{x} \text{ (for all vectors } \mathbf{x}\text{)}$$

Summary: Vector Space

- Notes:

(1) A vector space consists of four entities:

a set of vectors, a set of scalars, and two operations

$\text{V}:$ nonempty set

$c:$ scalar

$+(\mathbf{u},\mathbf{v})=\mathbf{u}+\mathbf{v}$ □ vector addition

$\bullet(c,\mathbf{u})=c\mathbf{u}$ □ scalar multiplication

$(V,\ +,\ \bullet)$ is called a vector space

(2) $V=\{\mathbf{0}\}$ □ zero vector space

Examples of vector spaces

(1) ***n*-tuple space: *R*^{n}**

The euclidean space $\mathbf{R}^n$, with the following operations

$$(u_1, u_2, \ldots u_n) + (v_1, v_2, \ldots v_2) = (u_1 + v_1, u_2 + v_2, \ldots u_n + v_n) \quad \text{vector addition}$$

$$k(u_1, u_2, \ldots u_n) = (ku_1, ku_2, \ldots ku_n) \quad \text{scalar multiplication}$$

satisfies all the rules outlined in Condition A & B, is the primary example of vector spaces.

Examples of vector spaces

(2) **Matrix space:** $V = M_{m\times n}$ (the set of all $m\times n$ matrices with real values)

Ex: $:(m = n = 2)$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ vector addition

$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ scalar multiplication

Vector Space (Contd.)

We are interested particularly in real vector spaces of real ***mx1*** column matrices. We denote such spaces by $\Re^m$, with vector addition and multiplication by scalars being as defined earlier for matrices. Vectors (column matrices) in $\Re^m$ are written as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

Vector Space (Contd.)

Example

The vector space with which we are most familiar is the two-dimensional real vector space $\Re^2$, in which we make frequent use of graphical representations for operations such as vector addition, subtraction, and multiplication by a scalar. For instance, consider the two vectors

$$\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Using the rules of matrix addition and subtraction we have

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \mathbf{a} - \mathbf{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Vector Space (Contd.)

Example (Con’t)

The following figure shows the familiar graphical representation of the preceding vector operations, as well as multiplication of vector **a** by scalar $c = -0.5$.

Subspaces of Vector Spaces

Consider two real vector spaces *W* and *V* such that:

- Each element of ***W*** is also an element of ***V*** (i.e., *W* is a *subset* of *V*).
- Operations on elements of *W* are the same as on elements of *V*. Under these conditions, *W* is said to be a ***subspace*** of *V*.
- **Note:** The set consisting of only **zero vector** in a vector space *V* is a subspace of V, called the **zero subspace** and written as **{0}**

Test for a subspace

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold.

1. The zero vector of V is in W.
2. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W.
3. If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

Summary: Subspaces of Vector Spaces

- **Subspace:**

$(V,+,\bullet)$: a vector space

$$\left.\begin{array}{l} W \neq \phi \\ W \subseteq V \end{array}\right\} \text{ : a nonempty subset}$$

$(W,+,\bullet)$: a vector space (under the operations of addition and scalar multiplication defined in V)

$\Rightarrow$ W is a subspace of V

- **Trivial subspace:**

Every vector space V has at least two subspaces.

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V.

(2) V is a subspace of V.

Test for a subspace: Example

- Let *W* be the set of all points in R^2 of the form **(3s, 2+5s).** Determine if W is a subspace of R^2.

Solution:

It can be seen that zero vector of R^2 is not in *W.* As if (3s,2+5s) were to be zero for some ***s***, then both *3s* and *2+5s* would be zero, which is impossible. Thus W is not a subspace of R^2.

Example 1: (The set of singular matrices is not a subspace of $M_{2\times 2}$)

Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2\times 2}$ with the standard operations.

Sol:

$$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\in W, B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\in W$$

$$\therefore A+B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\notin W$$

$\therefore W_2$ is not a subspace of $M_{2\times 2}$

- Example 2: The set of first-quadrant vectors is not a subspace of R^2

Show that $W = \{(x_1, x_2) : x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of R^2

Sol:

Let $\mathbf{u} = (1,\ 1) \in W$

$\Rightarrow\ (-1)\mathbf{u} = (-1)(1, 1) = (-1, -1) \notin W$

(W is not closed under scalar multiplication)

$\therefore W$ is not a subspace of R^2

Spanning Sets and Linear Independence

- Linear combination

A vector $\mathbf{u}$ in a vector space V is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ in V if $\mathbf{u}$ can be written in the form

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k,$$

where $c_1, c_2, \dots, c_k$ are real-number scalars

Example : Finding a linear combination

$\mathbf{v}_1 = (1,2,3) \quad \mathbf{v}_2 = (0,1,2) \quad \mathbf{v}_3 = (-1,0,1)$

Prove (a) $\mathbf{w} = (1,1,1)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

(b) $\mathbf{w} = (1,-2,2)$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Sol:

(a) $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$\begin{aligned}(1,1,1) &= c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1) \\ &= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3)\end{aligned}$$

$$\Rightarrow \begin{array}{rcrcrcl} c_1 & & & - & c_3 & = & 1 \\ 2c_1 & + & c_2 & & & = & 1 \\ 3c_1 & + & 2c_2 & + & c_3 & = & 1 \end{array}$$

Example : Finding a linear combination (Contd.)

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = 1 + t \,,\; c_2 = -1 - 2t \,,\; c_3 = t$$

(this system has infinitely many solutions)

$$\overset{t=1}{\Rightarrow} \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$$

$$\overset{t=2}{\Rightarrow} \mathbf{w} = 3\mathbf{v}_1 - 5\mathbf{v}_2 + 2\mathbf{v}_3$$

⊠

Example : Finding a linear combination (Contd.)

(b)

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$\Rightarrow$ This system has no solution since the third row means

$$0 \cdot c_1 + 0 \cdot c_2 + 0 \cdot c_3 = 7$$

$\Rightarrow$ $\mathbf{w}$ can not be expressed as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

Linear Combination (Contd.)

- If $v_1, \ldots, v_k$ are vectors in $\mathbf{R}^n$, *then the set of all possible linear combinations of* $v_1, \ldots, v_k$ is called the subset of $\mathbf{R}^n$ *spanned (or generated) by* $\{v_1, \ldots, v_k\}$ and is denoted by **span** $\{v_1, \ldots, v_k\}$

Span

- That is **Span {$\mathbf{v}_1, \ldots, \mathbf{v}_k$ }** is the collections of all vectors that can be written in the form $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_k\mathbf{v}_k$, *with* $x_1, \ldots x_k$ scalars (i.e. weights).

- For example, the following vectors are all in **Span {$\mathbf{v}_1, \mathbf{v}_2$}**

$$\boxed{v_1 + v_2, \qquad v_1 - v_2, \qquad 2v_1 + 7v_2, \qquad \frac{1}{3}v_1 + \frac{2}{3}v_2}$$

Example: 1

Let W be the set of all vectors of the form **(a - 3b, b – a, a, b),**

where ***a*** and ***b*** are arbitrary scalars. That is, let W = {(a - 3b, b – a, a, b): a and b in R}. Show that W is a subspace of R^4.

Solution: Writing the vectors in **W** as column vectors. Then an arbitrary vector in **W** has the form:

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{V_1} + b\underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{V_2}$$

The calculations shows that

W = Span{v_1,v_2}, where $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are the vectors indicated here. Thus ***W*** is a subspace of $\mathbf{R}^4$.

Example: 2

$$Let\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \ \textbf{and}\ \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

For what value(s) of h will **y** be in Span $\{V_1, V_2, V_3\}$

Solution: The vector **y** belongs to Span $\{V_1, V_2, V_3\}$, if and only if there exists scalars x_1, x_2, x_3 such that :

$$x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

Now this vector equation is equivalent to a system of three equations in three unknowns. The **augmented matrix** for the system is:

$$\begin{bmatrix} 1 & \mathbf{5} & \mathbf{-3} & \mathbf{-4} \\ -1 & \mathbf{-4} & \mathbf{1} & \mathbf{3} \\ -2 & \mathbf{-7} & \mathbf{0} & \mathbf{h} \end{bmatrix}$$

Example: 2

- If we row reduce the augmented matrix for the system, we find that:

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

- The system is consistent, if and only if there is no pivot in fourth column, i.e. ***h-5*** must be zero. Therefore, y is in span $\{V_1, V_2, V_3\}$, if and only if **h =5**.

Exercise: 1

$$Let\ \mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} -6 \\ 3 \\ -5 \end{bmatrix}, \ \textbf{and}\ \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

Determine if **b** be is in Span $\{V_1, V_2, V_3\}$

Sol: Yes, as row reducing the augmented matrix [A b] , where, A = $[V_1\ V_2\ V_3]$ yields that the system is **consistent independent** i.e. a solution does exist implying b is in Span $\{V_1, V_2, V_3\}$

$$\left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Exercise: 2

Determine if **b** is linear combination of the columns of matrix A.

$$Let\ \mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{-2} & \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{1} & \mathbf{8} \end{bmatrix} \ \mathbf{and\ b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$$

Sol: No, as row reducing the augmented matrix [A b] , where, A = [V_1 V_2 V_3] yields that the system is **inconsistent** i.e. a solution does not exist implying **b** is not in Span A= [V_1,V_2,V_3] or **b** is not a linear combination of the columns of matrix A.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

Spanning set of a vector space

- The span of a set: span(S)

  If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the **span of S** is the set of all linear combinations of the vectors in S,

$$\text{span}(S) = \left\{ c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \mid \forall c_i \in R \right\}$$

  (the set of all linear combinations of vectors in S)

- Definition of a spanning set of a vector space:

  If every vector in a given vector space V can be written as a linear combination of vectors in a set S, then S is called a **spanning set** of the vector space V

- Note: The above statement can be expressed as follows

$$\begin{aligned} & \operatorname{span}(S) = V \\ \Leftrightarrow\ & S \text{ spans (generates) } V \\ \Leftrightarrow\ & V \text{ is spanned (generated) by } S \\ \Leftrightarrow\ & S \text{ is a spanning set of } V \end{aligned}$$

- **Example:**

(a) The set $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ spans R^3 because any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be written as

$$\mathbf{u} = u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$$

Example : A spanning set for R^3

Show that the set $S = \{(1,2,3),(0,1,2),(-2,0,1)\}$ spans R^3

Sol:

We must determine whether an arbitrary vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be expressed as a linear combination of $\mathbf{v}_1$=(1,2,3), $\mathbf{v}_2$=(0,1,2), and $\mathbf{v}_3 = (-2,0,1)$

If $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \Rightarrow$
$$\begin{aligned} c_1 \qquad\quad - 2c_3 &= u_1 \\ 2c_1 + \;\; c_2 \qquad\quad &= u_2 \\ 3c_1 + 2c_2 + c_3 &= u_3 \end{aligned}$$

The above problem thus reduces to determining whether this system is consistent for all values of u_1, u_2, and u_3

$$\because \quad |A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$$

> ※ if A is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution ($\mathbf{x} = A^{-1}\mathbf{b}$) given any $\mathbf{b}$
>
> ※ a square matrix A is invertible (nonsingular) if and only if det $(A) \neq 0$

$\therefore\ A\mathbf{x} = \mathbf{u}$ has exactly one solution for every $\mathbf{u}$

$\Rightarrow \text{span}(S) = R^3$

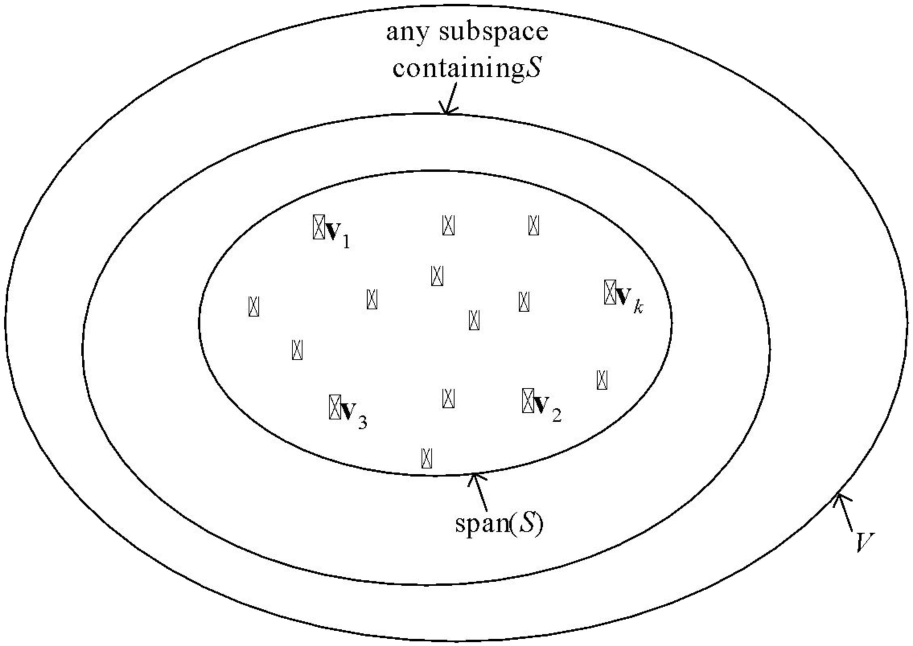
- Note:
  - For any set S_1 containing the set S, if S can span R^3, S_1 can span R^3 as well (e.g., $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \{(1,2,3), (0,1,2), (-2,0,1), (1,0,0)\}$).
  - Actually, in this case, what S_1 can span is only R^3. Since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ span R^3, $\mathbf{v}_4$ must be a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. So, adding $\mathbf{v}_4$ will not generate more combinations. As a consequence, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ only can span R^3

span(S) is a subspace of V

If $S=\{\mathbf{v}_1, \mathbf{v}_2,\ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then

(a) span(S) is a **subspace** of V

(b) span(S) is the smallest subspace of V that contains S, i.e., every other subspace of V containing S must contain span(S)



Proof:

(a)

Consider any two vectors **u** and **v** in span(S), that is,

$\mathbf{u}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k$ and $\mathbf{v}=d_1\mathbf{v}_1+d_2\mathbf{v}_2+\cdots+d_k\mathbf{v}_k$

Then $\mathbf{u}+\mathbf{v}=(c_1+d_1)\mathbf{v}_1+(c_2+d_2)\mathbf{v}_2+\cdots+(c_k+d_k)\mathbf{v}_k \in \text{span}(S)$ and $c\mathbf{u}=(cc_1)\mathbf{v}_1+(cc_2)\mathbf{v}_2+\cdots+(cc_k)\mathbf{v}_k \in \text{span}(S)$, because they can be written as linear combinations of vectors in S.

So, according to Thm. 4.5, we can conclude that span(S) is a subspace of V.

span(S) is a subspace of V :Example

Now example 1 of preceding section i.e.

For what value(s) of h will **y** be in Span $\{V_1, V_2, V_3\}$

$$Let\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{and}\ \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

Can be rewritten as follows:

- **For what value(s) of h will y be in subspace of R^3 Spanned by $\{V_1, V_2, V_3\}$**

Linear independence

Consider an $n \times n$ matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

This matrix can be viewed as an ordered set of column vectors: $\mathbf{A} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$

Where $v_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$ $v_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}$, etc.

Linear independence: Definition

A set of vectors is linearly dependent, if and only if we can find a set of scalars (i.e. weights) , $x_1, x_2, \cdots, x_n$ (not all of which are zero) such that

$$x_1 v_1 + x_2 x_2 + \cdots + x_n x_n = 0 --> (\text{vector equation of homogenous system})$$

If such a set of scalars cannot be found i.e. the vector equation has only trivial solution (i.e. all scalars are zero), then set of vectors $v_1, v_2, \cdots, v_n$ is said to be linearly independent.

Example 1: Testing for linear independence

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}=\{(1,\ 2,\ 3),(0,\ 1,\ 2),(-2,\ 0,\ 1)\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}\ \Rightarrow\ \begin{aligned} c_1 \qquad\quad -2c_3&=0\\ 2c_1+\ c_2+\qquad\ &=0\\ 3c_1+2c_2+\ c_3&=0\end{aligned}$$

$$\Rightarrow\left[\begin{array}{ccc|c}1&0&-2&0\\2&1&0&0\\3&2&1&0\end{array}\right]\xrightarrow{\text{G.-J. E.}}\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

$$\Rightarrow c_1=c_2=c_3=0\quad(\text{only the trivial solution})$$

$$(\text{or }\det(A)=-1\neq 0,\ \text{so there is only the trivial solution})$$

$\Rightarrow S$ is (or $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ are) linearly independent

Example 2: Testing for linear independence

- Determine whether the following set of vectors in the 2×2 matrix space is L.I. or L.D.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right\}$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$c_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example: 2 (Contd.)

$$\Rightarrow \quad \begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow c_1 = c_2 = c_3 = 0$ (This system has only the trivial solution)

$\Rightarrow S$ is linearly independent

- **A property of linearly dependent sets**

  A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_i$ in S can be written as a linear combination of the other vectors in S

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V are linearly dependent if and only if one is a scalar multiple of the other

Exercise: 1

$$Let \quad v_1 = \begin{bmatrix} 0 \\ -6 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -8 \\ -4 \\ 3 \end{bmatrix},$$

Determine if the set $\{v_1, v_2, v_3\}$ is linearly independent

Solution

We must determine if there is a nontrivial solution for given homogenous system using row operations on the associated **augmented matrix**

$$\left[\begin{array}{ccc|c} 0 & 0 & -8 & 0 \\ -6 & 4 & -4 & 0 \\ 1 & -2 & 3 & 0 \end{array}\right] \xrightarrow{R_1 \,\square\, R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -6 & 4 & -4 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right] \xrightarrow{6R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & -8 & 14 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right] \xrightarrow[-1/8\,R_3]{-1/8\,R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & -7/4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since, it can be seen from reduced echelon form that $x_1=x_2=x_3=0$ i.e. Trivial solution only, implying given set of vectors $\{v_1, v_2, v_3\}$ is **linearly independent.**

Exercise: 2

$$Let \quad v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 8 \\ 6 \\ 10 \end{bmatrix},$$

Determine if the set $\{v_1, v_2, v_3\}$ is linearly independent

Solution

Using row operations on the associated **augmented matrix** yields **row echelon matrix** given below

$$\boxed{\left[\begin{array}{ccc|c} 1 & -2 & 8 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}$$

x_1, x_2 : basic variables

x_3: free variable

Therefore, each non-zero value of x_3 determines a nontrivial solution of giver linear system. Hence, v_1,v_2,v_3 are **linearly dependent (i.e. not linearly independent).**

Exercise: 3

$$Let \quad v_1 = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix},$$

Determine if the set $\{v_1, v_2, v_3\}$ is linearly independent

Solution

Using row operations on the associated **augmented matrix** yields **row echelon matrix** given below

$$\left[\begin{array}{ccc|c} 1 & 1 & 1/3 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

x_1, x_2 , x_3 : basic variables

Since, it can be seen from row echelon form that this matrix can be row reduced to obtain $x_1=x_2=x_3=0$ i.e. Trivial solution only, implying given set of vectors $\{v_1, v_2, v_3\}$ is **linearly independent.**

Basis & Dimension

- A ***basis*** for a vector space *V* is a linearly independent spanning set for *V, i.e.* if a set of vectors **B = {v_1,v_2,…,v_n}** are linearly independent and span the vectorspace *V*, then the set B is called a ***basis*** *for the vectorspace V.*

- The number of vectors in the basis for a vector space is called the ***dimension*** of the vector space.

- If, for example, the number of vectors in the basis is *n*, we say that the vector space is *n*-dimensional.

Basis & Dimension (Contd.)

An important aspect of the concepts just discussed lies in the representation of any vector in $\mathfrak{R}^m$ as a ***linear combination*** of the basis vectors. For example, any vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

in $\mathfrak{R}^3$ can be represented as a linear combination of the basis vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Exercise: 1

$$Let\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}, \textbf{and } \mathbf{v}_5 = \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}$$

Find **bases** of Span $\{V_1, V_2, V_3, V_4, V_5\}$ i.e. find which of the columns in Span $\{V_1, V_2, V_3, V_4, V_5\}$ are **linearly independent**.

Sol: Simply writing the five column vectors as a matrix i.e., $A = [V_1\ V_2\ V_3\ V_4\ V_5]$ and row reducing the matrix :

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix}; \textbf{Row Echelon} = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise: 1 (Contd.)

- Row reducing the matrix to row echelon form yields that only $\{V_1, V_2, V_4\}$ are pivot columns i.e. linearly independent, indicating bases of Span $\{V_1, V_2, V_3, V_4, V_5\}$:

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}$$

Null space of the matrix (Nul A)

Consider the following system of homogenous equations:

$$x_1 - 3x_2 - 2x_3 = 0$$
$$-5x_1 + 9x_2 + x_3 = 0$$

In matrix form this system can be written as: $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$

- Recall that the set of all **x** that satisfy given homogenous system is called the ***solution set*** of the system.
- The *kernel* or *null space (also nullspace)* of a matrix A is the set of all vectors **x** for which **Ax = 0**.

Null space of the matrix: Example

$$Let\ \ A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}, \textbf{ and } \mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}.$$

Determine if u belongs to the null space of A.

To determine whether **u** belongs to **null space of A** or not, we simply need to check whether **Au = 0** or not.

$$Thus\ \ Au = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \textbf{Thus u is in Nul A.}$$

From above example, it can be seen that the null space of a matrix with **n** columns is a subspace of *n-dimensional* Euclidean space, as it is an essential requirement to define multiplication (Au).

Subspace properties

The null space of an $m \times n$ matrix is a subspace of R^n. That is, the set Null(A) has the following three properties:

- 1. Null(A) always contains the zero vector.
- 2. If $x \in$ Null(A) and $y \in$ Null(A), then **$x + y \in$ Null(A).**
- 3. If $x \in$ Null(A) and c is a scalar, then **$c\,x \in$ Null(A).**

Here are the proofs

- 1. $A0 = 0$.
- 2. If $Ax = 0$ and $Ay = 0$, then $A(x + y) = Ax + Ay = 0 + 0 = 0$.
- 3. If $Ax = 0$ and c is a scalar, then $A(cx) = cAx = c0 = 0$.

Null space of the matrix: Basis (contd.)

- The null space of a matrix is not affected by elementary row operations. This makes it possible to use *row reduction* to find a **basis** for the null space i.e.

- Input an ***m x n*** matrix A.
- Output a ***basis (linearly independent spanning set)*** *for the null space of A*

Procedure:

1. Use **elementary row operations** to put **A** in reduced row echelon form. For example, suppose that the reduced row echelon form of **A** is:

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 2 & -8 \\ 0 & 1 & 5 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & 7 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Null space of the matrix: Basis (contd.)

2. Interpreting the reduced row echelon form as a *homogeneous linear system*, determine which of the variables $x_1, x_2, \ldots, x_n$ are free. **Write equations** for the *dependent variables in terms of the free variables*. That is, the solutions to the homogeneous system given in in parametric form with ***x_3, x_5, and x_6*** as **free** variables are:

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 2 & -8 \\ 0 & 1 & 5 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & 7 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= 3x_3 - 2x_5 + 8x_6 \\ x_2 &= -5x_3 + x_5 - 4x_6 \\ x_4 &= -7x_5 + 9x_6. \end{aligned}$$

- Now solutions to the homogeneous system given in in parametric form with ***x_3, x_5, and x_6*** as **free variables** can be re-written into a *linear combination of vectors* where the weights are the free variables, that is:

$$\begin{aligned} x_1 &= 3x_3 - 2x_5 + 8x_6 \\ x_2 &= -5x_3 + x_5 - 4x_6 \\ x_4 &= \quad\quad - 7x_5 + 9x_6. \end{aligned} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} 3 \\ -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}} + x_5 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ -7 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}} + x_6 \underbrace{\begin{bmatrix} 8 \\ -4 \\ 0 \\ 9 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{w}}.$$

- For each *free variable x_i*, choosing the vector in the null space for which $x_i = 1$ and the remaining free variables are zero.

$$= x_3\mathbf{u} + x_5\mathbf{v} + x_6\mathbf{w}$$

- Note every linear combination of **u, v** and **w** is an **element of Nul A. Thus, {u,v,w}** is a ***linearly independent spanning set (basis)*** for Nul A.

Null space of the matrix: Basis (contd.)

3. The resulting collection of vectors is a **basis (linearly independent spanning set)** for the null space of A, i.e. the three vectors

$$\begin{bmatrix} 3 \\ -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ -7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ 0 \\ 9 \\ 0 \\ 1 \end{bmatrix}$$

are a *basis (spanning set)* for the null space of **A,** implying **Dimension of null space = 3**, as it contains 3 linearly independent vectors .

Null space of the matrix: Basis (contd.)

- **Bases are not unique**

The null space of a matrix **A** can have *several different bases*.

Now that we know that dimension of null space is 3, any three independent vectors from the null space of matrix A can serve as bases. For example; let x_3=1, x_5=x_6=0; & x_5=5, x_3=x_6=0; & x_6=2, x_3=x_5=0; (*remember, x_3, x_5 & x_6 are free variables*);

$$\left\{ \begin{bmatrix} 3 \\ -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ -7 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ 0 \\ 9 \\ 0 \\ 2 \end{bmatrix} \right\} \textit{with } x_3, x_5, x_6 \textit{ being free varaibles}$$

Exercise: 1

Find a linearly independent spanning set (basis) for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

basis (linearly independent spanning set) for the null space of A is consists of

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ with } x_2, x_4, x_5 \text{ being free varaibles}$$

Exercise: 2

Find a linearly independent spanning set (basis) for the null space of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 & 0 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 2 & 0 & 3 & -2 & 4 \end{bmatrix}$$

basis (linearly independent spanning set) for the null space of A is consists of

$$\left\{ \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 11/5 \\ 1/5 \\ -14/5 \\ 0 \\ 1 \end{bmatrix} \right\} \textit{with } x_4, x_5 \textit{ being free varaibles}$$

Column space of the matrix (Col *A*)

- The **column space** of a matrix (sometimes called the *range of a matrix*) is the set of all possible linear combinations of its column vectors.
- The column space of an $m \times n$ matrix A is a subspace of *m-dimensional Euclidean space*, as columns of matrix A each have **m** entries.
- The dimension of the column space is called the **rank** of the matrix.

- Let A be an $m \times n$ matrix, with column vectors $v_1, v_2, \ldots, v_n$. A linear combination of these vectors is any vector of the form:

$\mathbf{x_1v_1 + x_2v_2 + x_3v_3 + \ldots\ldots\ldots\ldots + x_nv_n}$; where $x_1, x_2, \ldots, x_n$ are scalars.

- The set of all possible linear combinations of $v_1,\ldots,v_n$ is called the **column space of A**. That is, the column space of A is the **span** of the vectors $\mathbf{v_1,\ldots,v_n}$.

$$Example:\ If\ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}, then\ the\ \ column\ vecrors\ are\ v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\ \&\ v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

A **linear combination** of $\mathbf{v}_1$ and $\mathbf{v}_2$ is any vector of the form

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 2x_1 \end{bmatrix}$$

The set of all such vectors is the **column space** of *A*.

Column space of the matrix (cont)

Any linear combination of the column vectors of a matrix ***A*** can be written as the **product** of *A* with a column vector, i.e.:

$$A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = x_1 v_1 + \dots\dots\dots + x_n v_n$$

- Therefore, the **column space of A** consists of all possible products Ax, for $x \in R^n$.
- This is the same as the ***image (or range)*** of the corresponding matrix transformation.

Column space of the matrix: Basis

- The columns of **A** span the column space, but they may not form a **basis** if the column vectors are not linearly independent.
- Since, elementary row operations do not affect the dependence relations between the column vectors. This makes it possible to use row reduction to **find a basis** for the column space.

- **Example:** Consider the matrix **A**, and find basis for the column space

$$A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 3 & 9 \\ 1 & 5 & 3 & 1 \\ 1 & 2 & 0 & 8 \end{bmatrix}.$$

Solution:

The columns of this matrix span the column space, but they may not be linearly independent, in which case **some subset** of them **will form a basis**. To find this basis, we reduce A to reduced row echelon form:

$$\begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 3 & 9 \\ 1 & 5 & 3 & 1 \\ 1 & 2 & 0 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & -3 \\ 0 & -1 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

At this point, it is clear that the first, second, and fourth columns are linearly independent, while the **third column is linearly dependent** as it is a linear combination of the first two. (Specifically, $v_3 = -2v_1 + v_2$.)

Example (contd.)

Therefore, the first, second, and fourth columns of the original matrix are a **basis** *for the column space*:

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 7 \\ 5 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 9 \\ 1 \\ 8 \end{bmatrix}.$$

- Note that the independent columns of the reduced row echelon form are precisely the columns with **pivots**.
- This makes it possible to determine which columns are linearly independent by reducing only to **echelon form**.

Exercise: 1

Find a linearly independent spanning set (basis) for the column space of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{-2} & \mathbf{2} \\ \mathbf{2} & \mathbf{0} & \mathbf{3} & \mathbf{-2} & \mathbf{4} \end{bmatrix}$$

basis (linearly independent spanning set) for the column space of A is consists of

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -\mathbf{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\} \textit{with } v_1, v_2, v_3 \textit{ being linearly independent}$$

Contrast between Nul A & Col A

- The **null space** and **column space** of a matrix are quite dissimilar, as subsequent examples will show. However, there is a connection between them, that will be studied later on.

$$EXAMPLE: \quad Let\ A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

a. If the null space of A is the subspace of R^k, what is K?

b. If the column space of A is the subspace of R^k, what is K?

- **Solution (a):** A vector **x** such that Ax is defined must have four entries, so Nul A is subspace of R^k when **k = 4**
- **Solution (b):** The columns of **A** each have three entries, so Col A is a subspace of R^k when **k = 3**

Exercise: 1

$$EXAMPLE: \quad Let\ A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$$

a. If the null space of A is the subspace of R^k, what is K?

b. If the column space of A is the subspace of R^k, what is K?

- **Solution (a):** A vector **x** such that Ax is defined must have two entries, so Nul A is subspace of R^k when **k = 2**
- **Solution (b):** The columns of **A** each have four entries, so Col A is a subspace of R^k when **k = 4**

Contrast between Nul A & Col A: Example 2

Find a nonzero vector in Nul A and a nonzero vector in Col A.

$$Let\ A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

- **Solution (a):** To find a nonzero vector in Nul A, we have to do some work i.e. row reduce the augmented matrix [A 0] to obtain

$$\begin{bmatrix} A & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} => x_1 = -9x_3; x_2 = 5x_3 \ \& \ x_4 = 0$$

- **Since, x_3 is a free variable, so assigning $x_3 = 1$, yields a vector in Nul A, namely**

$$\begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$

Example: 2 (contd.)

- **Solution (b):** It easier to find a nonzero vector in Col A, as any column of A will do, say column 1 i.e. **[2 -2 3]^t or** $\boxed{\begin{bmatrix} 2 \\ -\mathbf{2} \\ 3 \end{bmatrix}}$

Exercise: 2

Find a nonzero vector in Nul A and a nonzero vector in Col A.

$$Let\ A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$$

- **Solution (a):** To find a nonzero vector in Nul A, we have to do some work i.e. row reduce the augmented matrix [A 0] to obtain

$$\begin{bmatrix} A & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = 3x_2$$

- **Since, x_2 is a free variable, so assigning $x_2 = 1$, yields a vector in Nul A, namely**

$$\boxed{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$

Contrast between Nul A & Col A: Example 3

$$Let\ A = \begin{bmatrix} 2 & \mathbf{4} & \mathbf{-2} & \mathbf{1} \\ -2 & \mathbf{-5} & \mathbf{7} & \mathbf{3} \\ 3 & \mathbf{7} & \mathbf{-8} & \mathbf{6} \end{bmatrix}, \mathbf{u} = \begin{bmatrix} \mathbf{3} \\ \mathbf{-2} \\ \mathbf{-1} \\ \mathbf{0} \end{bmatrix} \textbf{ and } \mathbf{v} = \begin{bmatrix} \mathbf{3} \\ \mathbf{-1} \\ \mathbf{3} \end{bmatrix}$$

a. Determine if **u** is in null space of A (Nul A). Could **u** be in column space of A (Col A) ?

b. Determine if **v** is in column space of A (Col A). Could **v** be in null space of A (Nul A) ?

- **Solution (a):** To determine whether **u** belongs to **null space of A** or not, we simply need to check whether **Au = 0** or not.

$$Au = \begin{bmatrix} 2 & \mathbf{4} & \mathbf{-2} & \mathbf{1} \\ -2 & \mathbf{-5} & \mathbf{7} & \mathbf{3} \\ 3 & \mathbf{7} & \mathbf{-8} & \mathbf{6} \end{bmatrix} \begin{bmatrix} \mathbf{3} \\ \mathbf{-2} \\ \mathbf{-1} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{-3} \\ \mathbf{3} \end{bmatrix} \neq \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

So, **u** is not in Nul A, and with four coordinates **u** could not possibly be in Col A, since Col A is a subspace of R^3.

Example: 3 (contd.)

- **Solution (B): Since, a typical vector v in Col A has the property that equation Ax = v is consistent.**

- **Therefore,** to determine whether **v** belongs to **Col space of A or not,** we simply reduce [A   v] to an echelon form.

$$\begin{bmatrix} A & \mathbf{v} \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

- It is clear that the equation **Ax = v** is consistent, so it can be concluded that **v** is in Col A.

- However, with only three coordinates **v** *could not possibly in Nul A*, as Nul A is a subspace of R^4.

Exercise: 1

Find a spanning set (basis) for the null space & column space (i.e. range) of the matrix A

$$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 5 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

basis (linearly independent spanning set) for the null space of A is consists of

$$with \quad x_2 \quad \& \quad x_4 \; being\ free\ variable \qquad \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Exercise: 1 (contd.)

basis (linearly independent spanning set) for the column space (or range) of A is consists of

$$a \quad basis \quad for \quad the \quad range \quad of\ A = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \\ 1 \end{pmatrix} \right\}$$

It can be checked that ***A*** is not invertible i.e. is singular.

Because the determinant for such matrix should be 0.

$$\det \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 1 & 3 & 5 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0$$

Exercise: 2

Find a spanning set (basis) for the null space & column space (i.e. range) of the matrix A

$$A = \begin{pmatrix} 6 & -10 & 3 \\ 4 & -7 & 2 \\ 2 & -4 & 1 \end{pmatrix}$$

basis (linearly independent spanning set) for the null space of A is consists of

$$\left(\begin{array}{ccc|c} 6 & -10 & 3 & 0 \\ 4 & -7 & 2 & 0 \\ 2 & -4 & 1 & 0 \end{array}\right) \quad \textit{reduces} \quad \textit{to} \quad \left(\begin{array}{ccc|c} 1 & 0 & \tfrac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{aligned} x &= -\tfrac{1}{2}z \\ y &= 0 \\ z &= z \end{aligned}$$

A basis for the null space is: $\left\{ \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}$

Exercise: 2 (contd.)

basis (linearly independent spanning set) for the column space (or range) of A is consists of

$$B=\left\{\begin{pmatrix}-10\\-7\\4\end{pmatrix},\begin{pmatrix}3\\2\\1\end{pmatrix}\right\}$$

It can be checked that *A* is not invertible i.e. is singular.

Because the determinant for such matrix should be 0.

$$\det A=\begin{vmatrix}6 & -10 & 3\\4 & -7 & 2\\2 & -4 & 1\end{vmatrix}=0$$

Exercise: 3

Find a spanning set (basis) for the null space & column space (i.e. range) of the matrix A

$$A = \begin{bmatrix} -2 & \mathbf{4} & \mathbf{-2} & \mathbf{-4} \\ 2 & \mathbf{-6} & \mathbf{-3} & \mathbf{1} \\ -3 & \mathbf{8} & \mathbf{2} & \mathbf{-3} \end{bmatrix}$$

basis (linearly independent spanning set) for the null space of A is consists of

A basis for the null space is: $\left\{ \begin{pmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{pmatrix} \right\}$

Exercise: 3 (contd.)

basis (linearly independent spanning set) for the column space (or range) of A is consists of

$$B = \left\{ \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ -6 \\ 8 \end{pmatrix} \right\}$$

Row space of the matrix (Row A)

- The **row space** of a matrix is the set of all possible linear combinations of its row vectors.
- The row space of an $m \times n$ matrix A is a subspace of *n-dimensional Euclidean space*, as rows of matrix A each have **n** entries.
- The dimension of the row space is called the **rank** of the matrix.

- Let A be an $m \times n$ matrix, with row vectors $r_1, r_2, \ldots, r_m$. A linear combination of these vectors is any vector of the form:

$$x_1 r_1 + x_2 r_2 + x_3 r_3 + \ldots\ldots\ldots\ldots + x_m r_m \quad ; \text{where } x_1, x_2, \ldots, x_m \text{ are scalars.}$$

- The set of all possible linear combinations of $r_1,\ldots,r_m$ is called the **row space of A**. That is, the column space of A is the **span** of the vectors **$r_1,\ldots,r_m$**.

$$\textit{Example: If } A = \begin{bmatrix} 1 & 0 & \mathbf{2} \\ 0 & 1 & \mathbf{0} \end{bmatrix}, \textit{then the } \textbf{row} \textit{ vecrors are } r_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}^{\mathbf{T}} \;\&\; r_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^{T}$$

A **linear combination** of $\mathbf{r}_1$ and $\mathbf{r}_2$ is any vector of the form

$$x_1(1,0,2) + x_2(0,1,0) = (x_1, x_2, 2x_1)$$

The set of all such vectors is the **row space** of *A*.

Row space of the matrix : Basis

The row space is not affected by elementary row operations. This makes it possible to use row reduction *to find a basis* for the row space.

For example, consider the matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 7 & 4 \\ 1 & 5 & 2 \end{bmatrix}.$

The rows of this matrix span the row space, but they may not be linearly independent, in which case the rows will not be a basis. To find a basis, we reduce A to row echelon form, $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ represents the rows

Row space of the matrix : Basis (contd.)

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 7 & 4 \\ 1 & 5 & 2 \end{bmatrix} \underbrace{\sim}_{r_2 - 2r_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 1 & 5 & 2 \end{bmatrix} \underbrace{\sim}_{r_3 - r_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \underbrace{\sim}_{r_3 - 2r_2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Once the matrix is in echelon form, the **nonzero rows are a basis for the row space**.
- In this case, the **basis** is **{ (1, 3, 2), (0, 1, 0) }.**
- Another **possible basis** **{ (1, 0, 2), (0, 1, 0) }** comes from a further reduction.
- The **dimension** of the row space is called the **rank of the matrix**. This is the same as the **maximum number of linearly independent rows** that can be chosen from the matrix. For example, the 3 x 3 matrix in the example above has **rank two**.

Exercise: 1

Find bases for the row space, column space & null space of the matrix A

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

To find bases for the row space and the column space, row reducing A to row echelon form

$$A_{row\ echelon} = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise: 1 (Contd.)

Once, A is reduced to row echelon form, the non zero rows are bases for the row space . Thus;

Bases for Row (A) ={(1,3,-5,1,5), (0,1,-2,2,-7), (0,0,0,-4,20)}

For the column space, it can be seen that pivots are in Column 1, 2 and 4 of row echelon form;

Basis for Col (A) = $\left\{ \begin{pmatrix} -2 \\ 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ -3 \\ 11 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 7 \\ 5 \end{pmatrix} \right\}$

Exercise: 1 (Contd.)

For Nul (A) we need reduced row echelon form;

$$A_{reduced\ row\ echelon} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

basis (linearly independent spanning set) for the null space of A is consists of

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\} with\ x_3, x_5\ being\ free\ varaibles$$

Example *For the* 4×5 *matrix*

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix},$$

find bases for col(A), *nul*(A), *and row*(A).

Solution *From earlier work that we have done, we know how to find bases for col*(A) *and nul*(A).

Since

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the first, second, and fourth columns of A *are the pivot columns of* A. *We also know that the pivot columns of* A *form a basis for col*(A). *Thus, col*(A) *is three–dimensional and a basis for col*(A) *is*

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}.$$

Example (Contd.)

To find a basis for nul(A), we observe that every solution of $A\mathbf{x} = \mathbf{0}_4$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s-t \\ -3s+2t \\ t \\ 5s \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}.$$

Therefore, nul(A) is two–dimensional and a basis for nul(A) is

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

Another way of finding row (A)

To find a basis for $row(A)$*, we use the fact that* $row(A) = col(A^T)$*. Observing that*

$$A^T = \begin{bmatrix} -2 & 1 & 3 & 1 \\ -5 & 3 & 11 & 7 \\ 8 & -5 & -19 & -13 \\ 0 & 1 & 7 & 5 \\ -17 & 5 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the first, second, and fourth columns of A^T *are the pivot columns of* A^T*. Therefore,* $col(A^T) = row(A)$ *is three–dimensional and a basis for* $col(A^T) = row(A)$ *is*

$$\left\{ \begin{bmatrix} -2 \\ -5 \\ 8 \\ 0 \\ -17 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -13 \\ 5 \\ -3 \end{bmatrix} \right\}.$$

Summary

- The **null space** of an *m x n* matrix, **A**, denoted by nul (A) is the set of all solutions, x, of the equation **Ax = 0**.

- The **column space** of an *m x n* matrix, **A**, denoted by col (A) is the set of all **linear combinations** of the *column vectors* of **A**.

- The **row space** of an *m x n* matrix, **A**, denoted by row (A) is the set of all **linear combinations** of the *row vectors* of **A**.

Summary (Contd.)

Here are some basic observations about the row space, column space, and null space:

1. If A is an ***m x n*** matrix, then col (A) is a subspace of R^m and row (A) is a subspace of R^n. In particular, col (A) is the span of the columns of A and row (A) is the span of the rows of A.

2. If A is an ***m x n*** matrix, then

$$\mathbf{dim(col\ (A)) + dim(nul\ (A)) = n.}$$

n = total number of columns in A

3. If A is an m x n matrix, then row (A) = col (A^T) and col (A) = row(A^T).

Rank of a Matrix

The **rank** of a matrix is defined as

(a) **Maximum number of independent rows** in a matrix (i.e., rows that are not linear combinations of other rows) **or**

(b) **the maximum number of linearly independent column vectors** in the matrix

- If the rank of a **square matrix** is less than its dimension then the matrix is singular i.e. not invertible, implying there can be two cases:
  - Infinite number of solutions
  - No solutions

Example:1

The **rank** of a matrix **A**, written as *r*(**A**), is the **maximum number** of **linearly independent row vectors** of a matrix **A**.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Since; $R_2 = 2R_1$

Answer: $r(\mathbf{A}) = 2$

Rank: Facts (Contd.)

- **A** and its **transpose** $\mathbf{A}^T$ have the same rank.
- **Row-equivalent matrices** have the same rank.
- The rank of a matrix would be **zero** only if the matrix had **no elements**. If a matrix had even one element, its minimum rank would be one.
- For an ***m* x *n*** matrix,

  - **If *m* is less than *n***, then the maximum rank of the matrix is *m*.

  - If *m* is greater than *n*, then the maximum rank of the matrix is *n*.

Examples

1. **If *A* is *7 x 5* matrix what is the largest possible rank of A ?**

   Solution: Since A is *7 x 5* i.e. number of rows (m = 7) are greater than number of columns (n = 5), so maximum rank of matrix can only be equal to n i.e. *5*.

2. **If *A* is *5 x 7* matrix what is the largest possible rank of A ?**

   Solution: Since A is *5 x 7* i.e. number of rows (m = 5) are less than number of columns (n = 7), so maximum rank of matrix can only be equal to m i.e. *5*.

Example: 2

Compute the rank of matrix A

$$A = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 3 & 4 & 8 & 0 \end{bmatrix}$$

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4

- Since the matrix has more than zero elements, its rank must be greater than zero.
- And since it has **fewer rows than columns**, its maximum rank is equal to the maximum number of linearly independent rows.
- And because neither row is linearly dependent on the other row, the matrix has 2 linearly independent rows; so its **rank is 2**.

Example: 3

Compute the rank of matrix A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \\ 4 & 5 & 9 \end{bmatrix}$$

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

- Since the matrix has more than zero elements, its rank must be greater than zero. And since it has **fewer columns than rows**, its maximum rank is equal to the maximum number of linearly independent columns.

- **Columns 1 and 2 are independent**, because neither can be derived as a scalar multiple of the other. However, **column 3** is linearly dependent on columns 1 and 2, because column 3 is equal to column 1 plus column 2. That leaves the matrix with a maximum of two linearly independent columns; e.g., column 1 and column 2. So the matrix **rank is 2**.

Full Rank Matrices

- When all of the **vectors** in a matrix are linearly independent, the matrix is said to be **full rank**. Consider the matrices **A** and **B** below.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

- Notice that **row 2** of matrix **A** is a scalar multiple of row 1; that is, row 2 is equal to twice row 1.
- Therefore, rows 1 and 2 are linearly dependent.
- Matrix **A** has only one linearly independent row, so its rank is 1. Hence, matrix **A** is not full rank.

Full Rank Matrices (Contd.)

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

- Now, look at matrix B. All of its rows are linearly independent, so the **rank of matrix B is 3**.
- Matrix B is **full rank.**
- Note all **invertible i.e. non-singular** matrices will have full rank, as for such matrices all row or column vectors are linearly independent.

Computing Rank of a matrix

- The easiest way to compute the rank of a matrix A is given by the Gauss elimination method.
- The row-echelon form of A produced by the Gauss algorithm has the same rank as A, and
- Its **rank** can be read off as the number of non-zero rows or in other words number of pivot columns in A.

Example: 4

Consider for example the 4-by-4 matrix

$$A = \begin{bmatrix} 2 & 4 & 1 & 3 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 3 & 6 & 2 & 5 \end{bmatrix}.$$

It can be noted that:

- The **second column** is **twice** the first column, and
- that the **fourth column** equals the **sum** of the first and the third.
- Therefore; **first and the third** columns are linearly independent, so the **rank of *A*** is **two**.

Example: 4 (Contd.)

- This can be confirmed with the Gauss algorithm. It produces the following row echelon form of *A*:

$$A = \begin{bmatrix} 1 & \mathbf{2} & \mathbf{1/2} & \mathbf{3/2} \\ 0 & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Which has two non-zero rows, so the **rank of *A*** is **two**.

Exercise: 1

Compute the rank of matrix A

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix}$$

Rank of the matrix can be computed with the Gauss algorithm. It produces the following **row echelon form of *A***

$$A_{row\ echelon} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Which has two non-zero rows, so the **rank of *A*** is **two**.

Exercise: 1 (Contd.)

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix}$$

It can be noted that:

- Row 1 and Row 2 of matrix A are linearly independent.
- However, **Row 3** is a linear combination of Rows 1 and 2.

  Specifically, Row 3 = 3*(Row 1) + 2*(Row 2). .
- Therefore, matrix A has only **two independent row vectors**, so the **rank of *A*** is **two**.

Exercise: 2

Compute the rank of matrix A

$$A = \begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 & 0 & 1 \\ 0 & 4 & 2 & 4 & 3 & 3 \\ 1 & 3 & 2 & -2 & 0 & 0 \end{bmatrix}$$

Rank of the matrix can be computed with the Gauss algorithm.
It produces the following **row echelon form of *A***

$$A_{row\ echelon} = \begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Which has four non-zero rows, so the **rank of *A*** is **four**.

Rank Theorem

The **dimensions** **of the *column space* and the *row space* of a *m x n* matrix *A* are equal**. This common dimension i.e. rank of A , also equals the pivot positions in A and satisfies the equation:

rank A + dim Nul A = n

Here;

Dim null A (Dimension of null space or nullity) : Number of free variables in the equation Ax=0 or in other terms the number of columns of A that are not pivot columns.

n = number of columns of A

Examples

1. **If *A* is *7 x 9* matrix with a two dimensional null space, what is the rank of A ?**

Solution: Since A has 9 columns. Therefore, using Rank Theorem:

$$\text{rank } A + \dim \text{Nul } A = 9$$

$$\text{rank } A + 2 = 9$$

$$\text{rank } A = 7$$

- **Note that A is *7 x 9* i.e. number of rows (m = 7) are less than number of columns (n = 9), so maximum rank of matrix can only be equal to m i.e. 7.**

Examples (Contd.)

2. **Could a *6 x 9* matrix A have a two dimensional null space ?**

Solution: If A has a two dimensional null space, then according to Rank Theorem it would have to have a rank of 7. But the columns of A are in R^6, so maximum rank of matrix can not exceed 6. Thus, dim Nul A is at least 3, since A has 9 columns.

Examples (Contd.)

3. If A is a *6 x 8* matrix, what is the smallest possible dimensional null space i.e. dim Null A?

Solution: If Note that A is *6 x 8* i.e. number of rows (m = 6) are less than number of columns (n = 8), so maximum rank of matrix can only be equal to m i.e. 6. Since, smallest possible dimensional null space i.e. dim Null A will occur when rank is maximum i.e. equal to 6. Therefore, by Rank Theorem,

$$\text{Dim Null } A = 8 - 6 = 2$$

Example: 5

Compute rank and dimension of null space *(i.e. **nullity or dim Nul A**)* of matrix A.

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$$

Now performing **Gauss elimination** to obtain **Row echelon form**:

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \xrightarrow[-5R_1+R_3]{R_1+R_2} \begin{bmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 14 & -35 & 42 \end{bmatrix} \xrightarrow{7R_2+R_3} \begin{bmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since, number of non-zero rows are 2, so the **rank of *A*** is **2**.

Therefore, by Rank theorem, **dim Nul A = 4 - 2 = 2**

Example: 6

Compute rank and dimension of null space *(i.e. **nullity or dim Nul A**)* of matrix A.

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}$$

Now performing **Gauss elimination** to obtain **Row echelon form**:

$$\begin{array}{l} 2R_1+R_2 \\ 3R_1+R_3 \\ -3R_1+R_4 \end{array} \quad ===> \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 6 & -9 & 24 \\ 0 & 0 & -8 & 12 & -27 \end{bmatrix}$$

Non-pivot column

Example: 6 (contd.)

$$-3R_2+R_3 \quad 4R_2+R_4 \implies \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Since, rows of zeros should be at bottom of row echelon form

$$2R_1+R_2 \quad 3R_1+R_3 \quad -3R_1+R_4 \implies \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & \boxed{0} & 5 \\ 0 & 0 & 0 & \boxed{0} & 0 \end{bmatrix}$$

Non-pivot column

Since, number of non-zero rows are 1, so the **rank of *A*** is **3**.

Therefore, by Rank theorem, **dim Nul A = 5 - 3 = 2**

Applications

- One useful application of calculating the rank of a matrix is the **computation of the number of solutions** of a system of linear equations.

- A system of linear equations has a solution if the rank of the augmented matrix is equal to the rank of the coefficient matrix.

- **Why?**

Applications(contd.)

- Because the only way for a system to have no solutions is for the augmented matrix in echelon form to have as its last nonzero row

  [0 … 0 c] with $c \neq 0$.

- *But now the rank of the augmented* matrix is greater than the rank of the coefficient matrix.

- So if the ranks are equal, it means that this case i.e. inconsistent system cannot arise, and system must have at least one solution.

Example: Inconsistent System

Solve the linear system

$$3x_1 + x_2 - x_3 = 2$$
$$x_1 - x_2 + x_3 = 2$$
$$2x_1 - x_2 + x_3 = 6$$

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

- **At first, computing rank of coefficient matrix by reducing it to row echelon form:**

$$R_1 \Leftrightarrow R_2 \begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & -4 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[-4R_2+R_3]{R_2 \Leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since, number of non-zero rows are 2, so the **rank of *COEFFICIENT Matrix*** is **2**.

Example: Inconsistent System (contd.)

- **Now computing rank of augmented matrix by reducing it to row echelon form:**

$$R_1 \Leftrightarrow R_2 \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 3 & 1 & -1 & 2 \\ 2 & -1 & 1 & 6 \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 4 & -4 & -4 \\ 0 & 1 & -1 & 2 \end{array}\right] \xrightarrow[-4R_2+R_3]{R_2 \Leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

Since, number of non-zero rows are 3, so the **rank of *AUGMENTED Matrix*** is **3**.

But now the rank of the augmented matrix (i.e. 3) is greater than the rank of the coefficient matrix (i.e. 2), implying an inconsistent solution i.e. no solution.

Applications(contd.)

- If the rank of the augmented matrix is equal to the rank of the coefficient matrix, then

- The **solution is unique** if and only if the rank equals the number of variables. For example consider following system:

$$
\begin{aligned}
3x_1 + x_2 - x_3 &= 2 \\
x_1 - x_2 + x_3 &= 2 \\
2x_1 + 2x_2 + x_3 &= 6
\end{aligned}
$$

Row Echelon

Coefficient matrix

$$
\begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & -4 \\ 0 & 0 & 3 \end{bmatrix} ==> Rank = 3
$$

Applications(contd.)

$$A_{augm} = \left[\begin{array}{ccc|c} 3 & 1 & -1 & 2 \\ 1 & -1 & 1 & 2 \\ 2 & 2 & 1 & 6 \end{array}\right] \xrightarrow[\text{Augmented matrix}]{\text{Row Echelon}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] => Rank = 3$$

- Note that row echelon form has no non-zero rows, so the **rank of *A*** is **three,** **implying** rank equals the number of variables i.e. there are **no free variables**.

Applications(contd.)

- If the rank of the augmented matrix is equal to the rank of the coefficient matrix, then the **solution is unique** if and only if the rank equals the number of variables.

- Otherwise the general solution has ***k*** free variables where ***k*** is the difference between the number of variables and the rank. **For example** consider following system

$$\begin{aligned} 2x_1 + \; 4x_1 - \; 2x_3 &= 0 \\ 3x_1 + \; 5x_2 \qquad\qquad &= 1 \end{aligned}$$

Applications(contd.)

$$\begin{aligned} 2x_1 + \ 4x_1 - \ 2x_3 &= 0 \\ 3x_1 + \ 5x_2 \qquad\quad &= 1 \end{aligned} \xrightarrow[\text{Coefficient matrix}]{\text{Row Echelon}} \begin{bmatrix} 1 & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-3} \end{bmatrix} ==> Rank = 2$$

$$A_{augm} = \left[\begin{array}{ccc|c} 2 & \mathbf{4} & \mathbf{-2} & \mathbf{0} \\ \mathbf{3} & \mathbf{5} & \mathbf{0} & \mathbf{1} \end{array}\right] \xrightarrow[\text{Augmented matrix}]{\text{Row Echelon}} \left[\begin{array}{ccc|c} 1 & \mathbf{2} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{-3} & \mathbf{-1} \end{array}\right] => Rank = 2$$

- Note that row echelon form of both coefficient & augmented matrix have two non-zero rows, so the **rank of *A*** is **two, implying** difference between the number of variables and the rank is **1**. i.e. there is one **free variable,** namely x_3.

Summary: Rank of matrix

- Defined as **number of linearly independent** rows (or columns)
- Matrix is **full rank** if all rows (columns) linearly independent
- Rank is **reduced** if rows are linearly dependent
- **Quick check** of rank of matrix, calculate determinant, if 0, it implies linear dependence, then matrix is not of full rank

Equivalent Statements

If **A** is a *n x n* *invertible* matrix, then

a) The column vectors of A are linearly independent.
b) The row vectors of A are linearly independent.
c) The column vectors of A span R^n.
d) The row vectors of A span R^n.
e) The column vectors of A form a basis for R^n.
f) The row vectors of A form a basis for R^n.
p) A has rank n.
q) A has nullity 0.